Notes on Selected topics to accompany Sakurai's "Modern Quantum Mechanics"

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Paradoxes of a classical electron

According to classical physics, electric current flowing through a circular loop produces a magnetic moment, $\vec{\mu}_{\rm mag}$. This quantity is directly proportional to an orbital angular momentum due to the circular motion of the charged particles.

$$\vec{\mu}_{\rm mag} = \frac{q}{2m}\vec{L}_{\rm orb}, \tag{1}$$

where q is the total charge, and m is the total mass of all charged particles moving in the loop. The ratio $|\vec{\mu}_{\rm mag}|/|\vec{L}_{\rm orb}| = q/(2m)$ is the gyromagnetic ratio, denoted by γ.

In classical physics, for any object which has the same charge density as mass density, the value of γ is $q/(2m)$. For example, this relation is valid for a spherical shell of charge (q) and mass (m) uniformly distributed on the surface. In this case, the densities are $\rho_{\rm charge} = q/(4\pi r^2)$ and $\rho_{\rm mass} = m/(4\pi r^2)$. Similarly, this relation is also valid for a solid sphere with uniform charge and mass density. For this case, we have $\rho_{\rm charge} = 3q/(4\pi r^3)$ and $\rho_{\rm mass} = 3m/(4\pi r^3)$. In the following, we will see that

1. the spin angular momentum, $\vec{s}$, (we will use small letter s for a single particle and captial letter S for many particles) of an electron cannot be understood as angular momentum arising from rotational motion of a charged object.

2. the gyromagnetic ratio of an electron (more specifically, its g-factor) cannot be understood as a deviation in its charge and mass densities. We cannot think of the charge of an electron to be uniformly distributed in a spherical shell or a solid sphere.

We will see that both assumptions can result in paradoxes.

What will be the radius of an electron if it is a uniformly charged classical spherical shell?

Let us imagine a classical electron to be a spherical shell (i.e. a hollow sphere) with radius r_e and a total charge of $q = -e$ uniformly distributed on its surface. The charge density is $\rho = q/(4\pi r_e^2)$. The potential energy (V) of this system arises entirely from electrostatistics. This energy is defined as the work done in bringing the charge q from infinity to the surface. From classical electrodynamics, we note this energy as

$$V = \frac{1}{2}\frac{q^2}{C} \tag{2}$$

where C is the capacity of the shell which can be derived from Gauss' law as

$$C = 4\pi r_e \varepsilon_0 \tag{3}$$

Now the potential energy is

$$V = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 r_e} \tag{4}$$

We can think of this energy as the maximum amount of energy a classical electron can have. Then, according to special relativity,

$$\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r_e} < m_e c^2 \tag{5}$$

where m_e is the mass of the electron $\left(9.11 \times 10^{-31}\,\text{kg}\right)$. We can now deduce the lower bound of the radius as

$$r_e > \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2m_e c^2} = 1.4 \times 10^{-15}\,\text{m}. \tag{6}$$

Thus if an electron is thought of as a classical spherical shell with uniform surface charge density, its radius must be equal to or larger than 1.4×10^{-15} m. This value is larger than the latest experimental value of radius of a proton, which is about 0.87×10^{-15} m! Repeating the classical mechanical exercise for a proton, with mass $m_p = 1836 m_e$ results in the classical proton radius of 0.77×10^{-18} m which clearly contradicts the widely recognized experimental value of r_p. In a future lecture, we will discuss more on r_p and its connection to the Lamb shift of H-atom line spectrum. The radius of an electron has not been determined experimentally so far. The present understanding is that electrons are point masses, i.e. particles with no spatial extent.

In quantum mechanics, we will not worry about the radius or the volume of an electron. Instead, we will be concerned with the probability to find an electron in a given region of space via probability amplitudes (i.e. wavefunctions) in various scenarios. To find the probability to find an electron in a region of $dxdydz$ around a position vector $\vec{r} = (x, y, z)$, we will evaluate

$$|\psi(\vec{r})|^2 dxdydz \tag{7}$$

and denote $|\psi(\vec{r})|^2$ as the probability density. We will also see that the total probability to find the electron should be 1

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\, |\psi(\vec{r})|^2 = 1 \tag{8}$$

What will be the speed of the equatorial region of an electron if it is a classically spinning sphere?

If the spin $\vec{s}$, of the electron is treated as classical spin, i.e., as mechanical angular momentum ($\vec{L}$), the magnitude of spin is given by

$$|\vec{s}| = L = I\omega \tag{9}$$

where I is the moment of inertia of the spherical shell, $I = (2/3)m_e r_e^2$. Note, for a solid sphere $I = (2/5)m_e r_e^2$. The speed at the equator (where the speed is maximum) of this classically spinning sphere can be calculated using the relation $v^{\text{equator}} = r_e\omega$.

$$|\vec{s}| = \frac{2}{3} m_e r_e^2 \frac{v^{\text{equator}}}{r_e} \tag{10}$$

$$\Rightarrow v^{\text{equator}} = \frac{3}{2} \frac{|\vec{s}|}{m_e r_e} \tag{11}$$

We can use the value $|\vec{s}| = \sqrt{s(s+1)}\hbar$ with $s = 1/2$ to get $v^{\text{equator}} = 1074 \times 10^8\ \text{m/s} = 358c$. This value is over two orders of magnitude larger than the speed of light, hence violating the special theory of relativity!

Later we will see that the spin of an electron is a quantum mechanical quantity, and there is no classical mechanical dynamical quantity that can be used to make an analogy. For the purpose of this course, and in most non-relativistic problems, we will find that the wavefunction of a fermion can be expressed as a product of a spatial wavefunction depending on the spatial coordinate ($\vec{r}$) of the fermion, and a spin wavefunction depending on the hitherto unknown spin variable, usually denoted by ω.

$$\psi(\vec{r}, \omega) = \phi(\vec{r})\, \chi(\omega) \tag{12}$$

The variable ω can take only discrete values in quantum mechanics. For a fermion with spin s the variable ω can take $2s+1$ values $-s, -s+1, \ldots, s-1, s$. Hence for an electron with $s = 1/2$, ω can take only two values $-1/2$ and $1/2$. We will introduce two spin functions, $\alpha(\omega)$ and $\beta(\omega)$ corresponding to spin up and spin down, respectively. The functions α and β form an orthonormal basis and satisfy the relations

$$\int d\omega\, \alpha^{\dagger}(\omega)\, \alpha(\omega) = \int d\omega\, \beta^{\dagger}(\omega)\, \beta(\omega) = 1 \tag{13}$$

and

$$\int d\omega\, \alpha^{\dagger}(\omega)\, \beta(\omega) = \int d\omega\, \beta^{\dagger}(\omega)\, \alpha(\omega) = 0 \tag{14}$$

Along with these relations, the wavefunction should satisfy the antisymmetry principle: A many electron wavefunction must be antisymmetric with respect to the interchange of the spatial and spin coordinates of any two electrons. More on this, later.