Notes on Selected topics to accompany Sakurai's "Modern Quantum Mechanics"

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Canonical Transformation

Suppose the state of a classical system is defined by the phase space coordinates $\{q_a, p_a\}$; $a = 1, 2, \ldots, N$. These coordinates satisfy the Hamilton's equations of motions

$$\dot{q}_a = \frac{\partial H}{\partial p_a},\ \dot{p}_a = -\frac{\partial H}{\partial q_a};\ a = 1, 2, \ldots, N$$

These equations can be derived using the variational method by solving

$$\delta \int_{t_1}^{t_2} dt \left[\sum_a p_a \dot{q}_a - H(q, p, t) \right] = 0.$$

Hamiltonian is defined as

$$H = \sum_a \frac{p_a^2}{2m_a} + V(q_a).$$

A canonical transformation is a mapping of the $2N$ variables $\{q_a, p_a\}$ to a different set of variables $\{Q_a, P_a\}$ for which there is another set of equations of motion which involves a different Hamiltonian H'

$$\dot{Q}_a = \frac{\partial H'}{\partial P_a},\ \dot{P}_a = -\frac{\partial H'}{\partial P_a};\ a = 1, 2, \ldots, N$$

These equations can also be derived using the variational method by solving

$$\delta \int_{t_1}^{t_2} dt \left[\sum_a P_a \dot{Q}_a - H'(Q, P, t) \right] = 0.$$

From both variational equations, we have

$$\delta \int_{t_1}^{t_2} dt \left[\sum_a p_a \dot{q}_a - H(q,p,t) \right] = \delta \int_{t_1}^{t_2} dt \left[\sum_a P_a \dot{Q}_a - H'(Q,P,t) \right].$$

For this equation to hold, we must have

$$\sum_a p_a \dot{q}_a - H(q,p,t) = \sum_a P_a \dot{Q}_a - H'(Q,P,t) + \frac{dF}{dt},$$

where F is called as the generation function of that canonical transformation. The generating function is defined by

$$\begin{aligned} \frac{dF}{dt} &= \sum_a \left(p_a \dot{q}_a - P_a \dot{Q}_a\right) + \left(H'(Q,P,t) - H(q,p,t)\right) \\ dF &= \sum_a \left(p_a \dot{q}_a - P_a \dot{Q}_a\right) dt + \left(H'(Q,P,t) - H(q,p,t)\right) dt \\ dF &= \sum_a \left(p_a dq_a - P_a dQ_a\right) + \left(H' - H\right) dt \end{aligned} \tag{1}$$

Type-1 canonical transformation

We can expand dF as an exact differential if F depends on old positions, new positions, and time $\{q_a, Q_a, t\}$.

$$dF_1 = \sum_{a=1} \left(\frac{\partial F_1}{\partial q_a} dq_a + \frac{\partial F_1}{\partial Q_a} dQ_a \right) + \frac{\partial F_1}{\partial t} dt. \tag{2}$$

Comparing Eq.(1) and Eq.(2), we find that

$$p_a = \frac{\partial F_1}{\partial q_a}, \quad P_a = -\frac{\partial F_1}{\partial Q_a}, \quad H' = H + \frac{\partial F_1}{\partial t}. \tag{3}$$

Type-2 canonical transformation

Let's rearrange Eq.(1), using $P_a dQ_a = d\left(P_a Q_a\right) - Q_a dP_a$ as

$$\begin{aligned} dF_1 &= \sum_a \left(p_a dq_a - d\left(P_a Q_a\right) + Q_a dP_a\right) + \left(H' - H\right) dt \\ dF_1 + d\sum_a \left(P_a Q_a\right) &= \sum_a \left(p_a dq_a + Q_a dP_a\right) + \left(H' - H\right) dt \\ dF_2 &= \sum_a \left(p_a dq_a + Q_a dP_a\right) + \left(H' - H\right) dt \end{aligned} \tag{4}$$

From this, we have F_2 to be a function of old positions, new momenta, and time $\{q_a, P_a, t\}$.

$$p_a = \frac{\partial F_2}{\partial q_a}, \quad Q_a = \frac{\partial F_2}{\partial P_a}, \quad H' = H + \frac{\partial F_2}{\partial t}. \tag{5}$$

Similarly, one can write down expressions for Type-3 (using variables $\{p_a, Q_a\}$) and Type-4 (using variables $\{p_a, P_a\}$) canonical transformations.

Identity transformation

As a special case of Type-2 canonical transformation, consider the identity transformation $q_a = Q_a$ and $p_a = P_a$, where the generating function (generator) is defined as $F_2 = \sum_a P_a q_a = I$.

Infinitesimal Translation as a canonical transformation

Suppose the system is a single particle with phase-space coordinates $q, p = x, p$. After infinitesimal translation, the new coordinates are $Q, P = x + dx, p$. Since the momentum remains the same, we will have same Hamiltonian before and after translation, $H' = H$. For the infinitesimal translation to be a Type-2 canonical transformation, according to Eq.(5), we should have

$$p = \frac{\partial F_2}{\partial x}, \quad x + dx = \frac{\partial F_2}{\partial p}$$

For this to hold, F_2 should be defined as

$$F_2 = (x + dx)p \tag{6}$$

Note that dx is a constant shift, so $\partial/\partial x(dx) = 0$. Since xp is the generating function of the identity transformation, we can write

$$F_2 = I + pdx \tag{7}$$

Relation to Quantum Mechanical Infinitesimal Translation

In Sakurai (1.6.4), we compare Eq.(7) to $\mathcal{T}(dx') = 1 - iKdx'$, which is Eq.(1.202), on page-41, in Edition-3. The term Kdx' should be dimensionless, so the dimension of K should be inverse of length, $[K] = L^{-1}$. We know that the dimension of momentum is given by $[p] = MLT^{-1}$, and using $E + \hbar\omega$, we get $[\hbar] = [E]\left[\omega^{-1}\right]$ or $[\hbar] = ML^2T^{-2}T$. So, we find $[p/\hbar] = L^{-1}$ having the same dimension as K. With all these information, Sakurai gives another perspective on the quantum mechanica infinitesimal translation operator as

$$\mathcal{T}(dx') = 1 - i\frac{p}{\hbar}dx' \tag{8}$$
