Notes on Selected topics to accompany Sakurai's "Modern Quantum Mechanics"

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Degeneracy Theorem and Wronskian

Theorem Bound states of one-dimensional quantum systems are non-degenerate. In the following, we will discuss a proof by contradiction. Before we see the proof, let's recall the definitions of a bound state and the mathematical concept of Wronskian.

Bound state

By bound state what we say is that the energy if the system is smaller than the asymptotic maximum of the potential, $E < \max[V(x)]$. In such cases, for the corresponding wavefunction to be normalizable, they wavefunction must vanish at values of x where is the potential is asymptotic. This is why, we say that one of the conditions for a 'well-behaved' wavefunction is that they vanish at infinity.

$$\lim_{x \to \pm\infty} \psi(x) = 0.$$

Wronskian

For any two functions, $f(x)$ and $g(x)$, their Wronskian ($W(f,g)$) is the determinant function defined as follows.

$$W(f,g) = \begin{vmatrix} f(x) & f^{(1)}(x) \\ g(x) & g^{(1)}(x) \end{vmatrix}$$

Note that $f^{(1)}(x)$ is just a short-hand notation for the first derivative of $f(x)$ with respect to x. It is easy to see that if $W(f,g) = 0$, then f and g are linearly dependent. If two vectors are linearly dependent, we mean that they are not orthogonal. In other words, the dot product of two linearly dependent vectors cannot be zero. For functions, instead of dot product, we should look at their inner product. We say that if f and g are linearly dependent, then $\int dx f^*(x)g(x) \neq 0$.

Let's verify this statement using a proof-by-contradiction. Let's assume that $f(x)$ and $g(x)$ are linearly independent (*i.e.*,$\int dx f^*(x)g(x) = 0$). Then, we will set $W(f,g) = 0$, to get a result that is contradictory to our assumption. Hence, we must conclude that if $W(f,g) = 0$, then $f(x)$ and $g(x)$ are linearly dependent.

$$
\begin{aligned}
W(f,g) &= 0 \\
\Rightarrow f(x)g^{(1)}(x) - g(x)f^{(1)}(x) &= 0 \\
\Rightarrow f(x)g^{(1)}(x) &= g(x)f^{(1)}(x) \\
\Rightarrow \frac{1}{g(x)}\frac{d}{dx}g(x) &= \frac{1}{f(x)}\frac{d}{dx}f(x) \\
\Rightarrow \int dx \frac{1}{g(x)}\frac{d}{dx}g(x) &= \int dx \frac{1}{f(x)}\frac{d}{dx}f(x) + C \\
\Rightarrow \int \frac{dg(x)}{g(x)} &= \int \frac{df(x)}{f(x)} + C \\
\Rightarrow \int d(\ln g(x)) &= \int d(\ln f(x)) + C \\
\Rightarrow \ln g(x) &= \ln f(x) + C \\
\Rightarrow \ln g(x) &= \ln f(x) + \ln D \\
\Rightarrow \ln g(x) &= \ln D f(x) \\
\Rightarrow g(x) &= D f(x)
\end{aligned}
$$

So, what we have found is that if $W(f,g) = 0$, then $f(x)$ and $g(x)$ are essentially the same function differing by a constant factor. If f and g are two wavefunctions (ψ_1 and ψ_x), and if their Wronskian vanishes, then these two wavefunctions only differ by a normalization constant!

Proof for degeneracy theorem

The degeneracy theorem states that 'bound states of one-dimensional quantum systems are non-degenerate.' Let us prove this statement, again, through a proof-by-contradiction. Let us assume that there are two degenerate wavefunctions, $\psi_1(x)$ and $\psi_2(x)$. These two wavefunctions are orthogonal (*i.e.*, linearly indepedent) but their energy expectation value is the same, $\langle\psi_1|\hat{H}|\psi_1\rangle = \langle\psi_2|\hat{H}|\psi_2\rangle = E$.

Let us write the Schrödinger equation for both the wavefunctions.

$$
\begin{aligned}
\psi_1^{(2)}(x) &= [V(x) - E]\,\psi_1(x) \\
\psi_2^{(2)}(x) &= [V(x) - E]\,\psi_2(x)
\end{aligned}
$$

We find that

$$\begin{aligned}
\frac{\psi_1^{(2)}(x)}{\psi_2^{(2)}(x)} &= \frac{\psi_1(x)}{\psi_2(x)} \\
\Rightarrow \psi_2(x)\psi_1^{(2)}(x) - \psi_1(x)\psi_2^{(2)}(x) &= 0 \\
\Rightarrow \psi_2(x)\psi_1^{(2)}(x) - \psi_1(x)\psi_2^{(2)}(x) + \psi_1^{(1)}(x)\psi_2^{(1)}(x) - \psi_1^{(1)}(x)\psi_2^{(1)}(x) &= 0 \\
\Rightarrow \left[\psi_2(x)\psi_1^{(2)}(x) + \psi_1^{(1)}(x)\psi_2^{(1)}(x)\right] - \left[\psi_1(x)\psi_2^{(2)}(x) + \psi_1^{(1)}(x)\psi_2^{(1)}(x)\right] &= 0 \\
\Rightarrow \frac{d}{dx}\left[\psi_2(x)\psi_1^{(1)}(x)\right] - \frac{d}{dx}\left[\psi_1(x)\psi_2^{(1)}(x)\right] &= 0 \\
\Rightarrow \frac{d}{dx}\left[\psi_2(x)\psi_1^{(1)}(x) - \psi_1(x)\psi_2^{(1)}(x)\right] = 0
\end{aligned}$$

So, we have

$$\psi_2(x)\psi_1^{(1)}(x) - \psi_1(x)\psi_2^{(1)}(x) = \text{constant}$$

The above equation holds for all values of x. In such cases, we can use the information available to us for some value of x. We have already seen that at infinity, the wave functions of bound states must vanish. So, can safely assume the constant in the above equation to be zero.

$$\psi_2(x)\psi_1^{(1)}(x) - \psi_1(x)\psi_2^{(1)}(x) = 0$$

The L.H.S. is nothing but the Wronskian, $W(\psi_1, \psi_2)$. Since it vanishes, the two wavefunctions must be proportional to one another. This means that both wavefunctions are same violating our assumption that there are two degenerate (*i.e.*, with the same energy expectation value) linearly-independent wavefunctions.